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The Hodge ring of Kähler manifolds

D. Kotschick and S. Schreieder

Dedicated to the memory of F. Hirzebruch

Abstract

We determine the structure of the Hodge ring, a natural object encoding the Hodge numbers of all compact Kähler manifolds. As a consequence of this structure, there are no unexpected relations among the Hodge numbers, and no essential differences between the Hodge numbers of smooth complex projective varieties and those of arbitrary Kähler manifolds. The consideration of certain natural ideals in the Hodge ring allows us to determine exactly which linear combinations of Hodge numbers are birationally invariant, and which are topological invariants. Combining the Hodge and unitary bordism rings, we are also able to treat linear combinations of Hodge and Chern numbers. In particular, this leads to a complete solution of a classical problem of Hirzebruch's.

1. Introduction

For the purpose of studying the spread and potential universal relations among the Betti numbers of manifolds, one can use elementary topological operations such as connected sums to modify the Betti numbers in examples. This leads to the conclusion that there are no universal relations among the Betti numbers, other than the ones imposed by Poincaré duality. However, not every set of Betti numbers compatible with Poincaré duality is actually realized by a (connected) manifold. This subtlety is removed, and the discussion in different dimensions combined into one, by the following definition: consider the Betti numbers as $\mathbb{Z}$-linear functionals on formal $\mathbb{Z}$-linear combinations of oriented equidimensional manifolds, and identify two such linear combinations if they have the same Betti numbers and dimensions. The quotient is a graded ring, the oriented Poincaré ring $\mathcal{P}_*$, graded by the dimension, with multiplication induced by the Cartesian product of manifolds. This ring has an interesting structure, which we determine in § 2 below. It turns out that $\mathcal{P}_*$ is finitely generated by manifolds of dimension at most 4, but is not a polynomial ring over $\mathbb{Z}$, although it does become a polynomial ring after tensoring with $\mathbb{Q}$.

In § 3 we carry out an analogous study for the Hodge numbers of compact Kähler manifolds. This is potentially much harder, since there is no connected sum or similar cut-and-paste operation in the Kähler category that would allow one to manipulate individual Hodge numbers $h^{p,q}$ in examples. Indeed, it seems to have been unknown until now, whether there are any universal relations among the Hodge numbers of Kähler manifolds beyond the symmetries $h^{q,p} = h^{p,q} = h^{n-p,n-q}$. Complex algebraic geometry does provide many constructions of Kähler manifolds, but these constructions are not as flexible as one might want them to be. Moreover,

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in spite of the recent work of Voisin [Voi10], the gap between complex projective varieties on the one hand and compact Kähler manifolds on the other is far from understood. We refer the reader to Simpson's thought-provoking survey [Sim04] for a description of the general state of ignorance concerning the spread of Hodge numbers and other invariants of Kähler manifolds.

It is our goal here to shed some light on the behaviour and properties of Hodge numbers of Kähler manifolds. For this purpose we consider the Hodge numbers as $\mathbb{Z}$-linear functionals on formal $\mathbb{Z}$-linear combinations of compact equidimensional Kähler manifolds and identify two such linear combinations if they have the same Hodge numbers and dimensions. The quotient is a graded ring, the Hodge ring $\mathcal{H}_*$, graded by the complex dimension, with multiplication again induced by the Cartesian product. Its structure is described by the following result.

Theorem 1. *The Hodge ring $\mathcal{H}_*$ is a polynomial ring over $\mathbb{Z}$, with two generators in degree* 1, *and one in degree* 2. *For the generators one may take the projective line $L=\mathbb{C}P^1$, an elliptic curve E, and any Kähler surface S with signature ± 1.*

Note that *a priori* it is not at all obvious that $\mathcal{H}_*$ is finitely generated, let alone generated by elements of small degree. Moreover, in the topological situation of the Poincaré ring, the corresponding structure is more complicated, in that $\mathcal{P}_*$ is not a polynomial ring over $\mathbb{Z}$.

The proof of this theorem has several important consequences, including the following.

(1) Since we may take the surface S to be projective, the Hodge ring is generated by projective varieties. This is in contrast with the work of Voisin [Voi10] on the Kodaira problem, which showed that more subtle features of Hodge theory do distinguish the topological types of projective manifolds from those of arbitrary Kähler manifolds.

(2) Counting monomials, we see that the degree n part $\mathcal{H}_n$ of the Hodge ring is a free $\mathbb{Z}$-module of rank equal to the number of Hodge numbers modulo the Kähler symmetries $h^{q,p}=h^{p,q}=h^{n-p,n-q}$. Thus there are no universal $\mathbb{Q}$-linear relations between the Hodge numbers, other than the ones forced by the known symmetries.

(3) The proof of Theorem 1 will show that the Hodge numbers $h^{p,q}$ with $0\leqslant q\leqslant p\leqslant n$ and $p+q\leqslant n$ form a $\mathbb{Z}$-module basis for $\mathrm{Hom}(\mathcal{H}_n,\mathbb{Z})$. Therefore, there are no non-trivial universal congruences among these Hodge numbers.

For technical reasons, we find it more convenient to work with a different definition of $\mathcal{H}_*$, rather than the one given above. However, it will follow from the discussion in § 3 below that the two definitions give the same result, and this fact will establish statement (3), cf. Remark 3.

In working with Hodge numbers, the Hodge ring plays a role analogous to that of the unitary bordism ring Ω^U_* in working with Chern numbers. This bordism ring is also generated by smooth complex projective varieties, and its structure shows that there are no universal $\mathbb{Q}$-linear relations between the Chern numbers, cf. § 6.1 below. However, in that case the analogue of statement (3) above is not true, in that there are universal congruences between the Chern numbers.

Our determination of the Hodge ring over $\mathbb{Z}$ allows us to write down all universal linear relations or congruences between the Hodge numbers of smooth projective varieties and their Pontryagin or Chern numbers.

(HP) A combination of Hodge numbers equals a combination of Pontryagin numbers if and only if it is a multiple of the signature, see Corollary 4.

(HC) A combination of Hodge numbers equals a combination of Chern numbers if and only if it is a combination of the $\chi_p=\sum_q(-1)^q h^{p,q}$, see Corollary 5.

In these statements the Hodge numbers are considered modulo the Kähler symmetries. We prove (HP) and (HC) in the strongest form possible, for equalities mod m for all m; the statements over $\mathbb{Z}$ or $\mathbb{Q}$ follow. While the validity of these relations is of course well known, their uniqueness is new, except that, with coefficients in $\mathbb{Q}$, statement (HC) could be deduced from [Kot12, Corollary 5]. Just as it was unknown until now whether there are universal relations between the Hodge numbers—we prove that there are none beyond the Kähler symmetries—their potential relations with the Chern and Pontryagin numbers were unknown.

In § 4 we analyse the comparison map $f\colon \mathcal{H}_* \longrightarrow \mathcal{P}_*$, whose image is naturally the Poincaré ring of Kähler manifolds. We will see that there are no universal relations between the Betti numbers of Kähler manifolds, other than the vanishing mod 2 of the odd-degree Betti numbers. Setting aside these trivial congruences, the only relations between the Betti numbers of smooth projective varieties and their Pontryagin or Chern numbers are the following.

(BC) A combination of Betti numbers equals a combination of Chern numbers if and only if it is a multiple of the Euler characteristic, see Corollary 7.

(BP) Any congruence between a $\mathbb{Z}$-linear combination of Betti numbers of smooth complex projective varieties of complex dimension $2n$ and a non-trivial combination of Pontryagin numbers is a consequence of $e \equiv (-1)^n\sigma \mod 4$, see Corollary 6. Here e and σ denote the Euler characteristic and the signature respectively.

In both statements the Betti numbers are considered modulo the symmetry imposed by Poincaré duality. In (BP) the conclusion is that there are no universal $\mathbb{Q}$-linear relations.

We shall determine several geometrically interesting ideals in the Hodge ring. An easy one to understand is the ideal generated by differences of birational smooth projective varieties. This leads to the following result, again modulo the Kähler symmetries of Hodge numbers.

Theorem 2. *The mod m reduction of an integral linear combination of Hodge numbers is a birational invariant of projective varieties if and only if the linear combination is congruent modulo m to a linear combination of the $h^{0,q}$.*

It follows that a rational linear combination of Hodge numbers is a birational invariant of smooth complex projective varieties if and only if, modulo the Kähler symmetries, it is a combination of the $h^{0,q}$ only.

Other ideals in $\mathcal{H}_*$ we will calculate are those of differences of homeomorphic or diffeomorphic complex projective varieties, thereby determining exactly which linear combinations of Hodge numbers are topological invariants. The question of the topological invariance of Hodge numbers was first raised by Hirzebruch in 1954. His problem list [Hir54] contains the following question about the Hodge and Chern numbers of smooth complex projective varieties, listed there as Problem 31.

Problem. Are the $h^{p,q}$ and the Chern characteristic numbers of an algebraic variety V_n topological invariants of V_n? If not, determine all those linear combinations of the $h^{p,q}$ and the Chern characteristic numbers which are topological invariants.

Since the time of Hirzebruch's problem list almost sixty years ago, this and related questions have been raised repeatedly in other places, such as a mathoverflow posting by S. Kovács in late 2010, asking whether the Hodge numbers of Kähler manifolds are diffeomorphism invariants. The special case of Hirzebruch's question where one considers linear combinations of Chern numbers only, without the Hodge numbers, was recently answered by the first author [Kot09, Kot12]. That answer used the structure results of Milnor [Mil60, Tho95] and Novikov [Nov62] for the unitary

bordism ring, exploiting the bordism invariance of Chern numbers. The Hodge numbers were not treated systematically in [Kot09, Kot12] because they are not bordism invariants. However, the results of those papers, and already of [Kot08], show that certain linear combinations of Hodge numbers that are bordism invariants because of the Hirzebruch–Riemann–Roch theorem are not (oriented) diffeomorphism invariants in complex dimensions $\geqslant 3$. This failure of diffeomorphism invariance of Hodge numbers, which can be traced to the fact that certain examples of pairs of algebraic surfaces with distinct Hodge numbers from [Kot92] become diffeomorphic after taking products with $\mathbb{C}P^1$, say, was also observed independently several years ago by F. Campana (unpublished).

In spite of these observations, the question of determining which linear combinations of Hodge numbers are topological invariants was still wide open. In § 5 below we settle this question using the Hodge ring and the forgetful comparison map $f\colon \mathcal{H}_* \longrightarrow \mathcal{P}_*$. The result is the following theorem.

Theorem 3. *The mod m reduction of an integral linear combination of Hodge numbers of smooth complex projective varieties is:*

(1) *an oriented homeomorphism or diffeomorphism invariant if and only if it is congruent mod m to a linear combination of the signature, the even-degree Betti numbers and the halves of the odd-degree Betti numbers; and*

(2) *an unoriented homeomorphism invariant in any dimension, or an unoriented diffeomorphism invariant in dimension $n \neq 2$, if and only if it is congruent mod m to a linear combination of the even-degree Betti numbers and the halves of the odd-degree Betti numbers.*

The corresponding result for rational linear combinations follows. Complex dimension 2 has to be excluded when discussing diffeomorphism invariant Hodge numbers, since in that dimension all the Hodge numbers are linear combinations of Betti numbers and the signature, and the signature is, unexpectedly, invariant under all diffeomorphisms, even if they are not assumed to preserve the orientation, see [Kot97, Theorem 6], and also [Kot08, Theorem 1].

In § 6 we consider arbitrary $\mathbb{Z}$-linear combinations of Hodge and Chern numbers. In the same way that the Hodge numbers lead to the definition of $\mathcal{H}_*$, these more general linear combinations lead to the definition of another ring, the Chern–Hodge ring $\mathcal{CH}_*$. We use $\mathcal{CH}_*$ to prove that Theorem 2 remains true for mixed linear combinations of Hodge and Chern numbers in place of just Hodge numbers. In this general setting, the conclusion of course has to be interpreted modulo the HRR relations, see Theorem 13 and Corollary 8, which also generalize a recent theorem about Chern numbers proved over $\mathbb{Q}$ by Rosenberg [Ros08, Theorem 4.2].

In § 7 we study certain ideals in $\mathcal{CH}_* \otimes \mathbb{Q}$, leading to the following answer to the general form of Hirzebruch's question, mixing the Hodge and Chern numbers in linear combinations.

Theorem 4. *A rational linear combination of Hodge and Chern numbers of smooth complex projective varieties is:*

(1) *an oriented homeomorphism or diffeomorphism invariant if and only if it reduces to a linear combination of the Betti and Pontryagin numbers after perhaps adding a suitable combination of the $\chi_p - \mathrm{Td}_p$; and*

(2) *an unoriented homeomorphism invariant in any dimension, or an unoriented diffeomorphism invariant in dimension $n \neq 2$, if and only if it reduces to a linear combination of the Betti numbers after perhaps adding a suitable combination of the $\chi_p - \mathrm{Td}_p$.*

As always, the Hodge numbers are considered modulo the Kähler symmetries. This theorem is a common generalization of Theorem 3 for the Hodge numbers and the main theorems of [Kot09, Kot12] for the Chern numbers. Once again complex dimension 2 has to be excluded in the statement about unoriented diffeomorphism invariants because the signature is a diffeomorphism invariant of algebraic surfaces by [Kot97, Theorem 6], see also [Kot08, Theorem 1]. We do not state this theorem for congruences, since we are unable to prove it if the modulus m is divisible by 2 or 3; compare Remark 5 in § 7.

Dedication

We dedicate this work to the memory of F. Hirzebruch, who first formulated the main problems treated here and who was one of the principal creators of their mathematical context. He read the preprint version of our solution, but sadly passed away before its publication in print. We are fortunate to have been influenced by him.

2. The Poincaré ring

In the introduction we defined the Poincaré ring by taking $\mathbb{Z}$-linear combinations of oriented equidimensional manifolds, and identifying two such linear combinations if they have the same Betti numbers and dimensions. Elements of this ring can be identified with their Poincaré polynomials

$$P_{t,z}(M) = (b_0(M) + b_1(M)\cdot t + \cdots + b_n(M)\cdot t^n)\cdot z^n \in \mathbb{Z}[t,z],$$

where the $b_i(M)$ are the real Betti numbers of M. Here we augment the usual Poincaré polynomial using an additional variable z in order to keep track of the dimension in linear combinations where the top-degree Betti number may well vanish. In this way we obtain an embedding of the Poincaré ring into $\mathbb{Z}[t,z]$. This embedding preserves the grading given by $\deg(t)=0$ and $\deg(z)=1$.

The Betti numbers satisfy the Poincaré duality relations

$$b_i(M) = b_{n-i}(M) \quad \text{for all } i \quad \text{and} \quad b_{n/2}(M) \equiv 0 \mod 2 \quad \text{if } n \equiv 2 \mod 4.$$

Not every polynomial having this symmetry and satisfying the obvious constraints $b_i(M)\geqslant 0$ and $b_0(M)=1$ can be realized by a connected manifold. For example, it is known classically that $(1+t^k+t^{2k})z^{2k}$ cannot be realized if k is not a power of 2; cf. [Hir53, § 2]. We sidestep this issue by modifying the definition of the Poincaré ring in the following way, replacing it by a potentially larger ring with a more straightforward definition.

Let $\mathcal{P}_n$ be the $\mathbb{Z}$-module of *all* formal augmented Poincaré polynomials

$$P_{t,z} = (b_0 + b_1\cdot t + \cdots + b_n\cdot t^n)\cdot z^n \in \mathbb{Z}[t,z],$$

satisfying the duality condition $b_i = b_{n-i}$ for all i and $b_{n/2}\equiv 0 \pmod 2$ if $n\equiv 2 \pmod 4$, regardless of whether they can be realized by manifolds. One could show directly that all elements of $\mathcal{P}_n$ are $\mathbb{Z}$-linear combinations of Poincaré polynomials of closed orientable n-manifolds, thereby proving that this definition of $\mathcal{P}_n$ coincides with the one given in the introduction. We will not do this here, but will reach the same conclusion later on, see Remark 1 below.

For future reference we note the following obvious statement.

Lemma 1. *The $\mathbb{Z}$-module $\mathcal{P}_n$ is free of rank $[(n+2)/2]$, spanned by the following basis:*

$$e^n_k = (t^k + t^{n-k})z^n \quad \text{for } 0\leqslant k < n/2,$$

and, if n is even,

$$e^n_{n/2} = t^{n/2}z^n \quad \text{if } n \equiv 0 \pmod 4,$$

respectively

$$e^n_{n/2} = 2t^{n/2}z^n \quad \text{if } n \equiv 2 \pmod 4.$$

We define the Poincaré ring by

$$\mathcal{P}_* = \bigoplus_{n=0}^{\infty} \mathcal{P}_n \subset \mathbb{Z}[t, z].$$

This is a graded ring whose addition and multiplication correspond to the disjoint union and the Cartesian product of manifolds, and the grading, induced by the degree in $\mathbb{Z}[t, z]$ with $\deg(t) = 0$ and $\deg(z) = 1$, corresponds to the dimension.

The structure of the Poincaré ring is completely described by the following theorem.

Theorem 5. *Let W, X, Y and Z have degrees 1, 2, 3 and 4 respectively. The oriented Poincaré ring $\mathcal{P}_*$ is isomorphic, as a graded ring, to the quotient of the polynomial ring $\mathbb{Z}[W, X, Y, Z]$ by the homogeneous ideal $\mathcal{I}$ generated by*

$$WX - 2Y, \quad X^2 - 4Z, \quad XY - 2WZ, \quad Y^2 - W^2Z.$$

Proof. Define a homomorphism of graded rings

$$P\colon \mathbb{Z}[W, X, Y, Z] \longrightarrow \mathcal{P}_*$$

by setting

$$P(W) = (1 + t)z, \quad P(X) = 2tz^2, \quad P(Y) = (t + t^2)z^3, \quad P(Z) = t^2z^4.$$

By definition, P vanishes on $\mathcal{I}$, and so induces a homomorphism from the quotient $\mathbb{Z}[W, X, Y, Z]/\mathcal{I}$ to $\mathcal{P}_*$. We will show that this induced homomorphism is an isomorphism. The first step is to prove surjectivity.

Lemma 2. *The homomorphism P is surjective.*

Proof. If $n \equiv 0 \pmod 4$, then $e^n_{n/2} = P(Z^{n/4})$. Similarly, if $n \equiv 2 \pmod 4$, then $e^n_{n/2} = P(XZ^{(n-2)/4})$. Thus we only have to prove that e^n_k is in the image of P for all $k < n/2$. We do this by induction on n.

It is easy to check explicitly that P is surjective in degrees $\leqslant 4$. Therefore, for the induction we fix some $n \geqslant 5$, and we assume that surjectivity of P is true in all degrees $< n$.

Consider first the case when n is even. Then for $k < n/2$ we have the following identity:

$$e^n_k = e^{n/2-k}_0 \cdot e^{n/2+k}_k - 2t^{n/2}z^n.$$

By the induction hypothesis the two factors $e^{n/2-k}_0$ and $e^{n/2+k}_k$ are in the image of P. Since we have already noted that $2t^{n/2}z^n$ is in the image of P, we conclude that P is surjective in degree n.

Finally, assume that n is odd. In this case we have

$$\begin{aligned} e^n_k &= (1 + t)z \cdot \left(\sum_{i=0}^{n-2k-1} (-1)^i t^{k+i} \right) z^{n-1} \\ &= (1 + t)z \cdot \left(\sum_{i \neq (n-2k-1)/2} (-1)^i t^{k+i} \right) z^{n-1} + (-1)^{(n-2k-1)/2}(1 + t)z \cdot t^{(n-1)/2}z^{n-1}. \end{aligned}$$

Here $(1+t)z = P(W)$ by definition, and the induction hypothesis tells us that

$$\left(\sum_{i \neq (n-2k-1)/2} (-1)^i t^{k+i} \right) z^{n-1}$$

with i running from 0 to $n-2k-1$ is in the image of P.

On the one hand, if $n \equiv 1 \pmod 4$, then $t^{(n-1)/2} z^{n-1} = P(Z^{(n-1)/4})$. On the other hand, if $n \equiv 3 \pmod 4$, then we rewrite

$$(1+t)z \cdot t^{(n-1)/2} z^{n-1} = (t+t^2) z^3 \cdot t^{(n-3)/2} z^{n-3} = P(YZ^{(n-3)/4}).$$

This completes the proof of surjectivity of P in all degrees. □

The next step in the proof of the theorem is to estimate the rank of the degree n part of the quotient $\mathbb{Z}[W, X, Y, Z]/\mathcal{I}$.

Lemma 3. *The degree n part of the quotient $\mathbb{Z}[W, X, Y, Z]/\mathcal{I}$ is generated as a $\mathbb{Z}$-module by at most $[(n+2)/2]$ elements.*

Proof. A generating set is provided by the images of the monomials $W^i X^j Y^k Z^l$ with $i + 2j + 3k + 4l = n$. The relations $X^2 = 4Z$ and $Y^2 = W^2 Z$ from the definition of $\mathcal{I}$ mean that we only have to consider $j = 0$ or 1 and $k = 0$ or 1. Further, since $XY = 2WZ$, we do not need any monomials where $j = k = 1$. Finally, since $WX = 2Y$, we may assume $i = 0$ whenever $j = 1$. Thus, a generating set for the degree n part of the quotient $\mathbb{Z}[W, X, Y, Z]/\mathcal{I}$ is given by the images of the monomials $W^i Z^l$, XZ^l and $W^i Y Z^l$.

Assume first that $n-2$ is not divisible by 4. In this case there is no monomial of the form XZ^l of degree n. The number of monomials of the form $W^i Z^l$ is $[(n+4)/4]$, and the number of monomials of the form $W^i Y Z^l$ is $[(n+1)/4]$. The sum of these two numbers is $[(n+2)/2]$, since we assumed that n is not congruent to 2 modulo 4.

If $n \equiv 2 \pmod 4$, then there is exactly one monomial of the form XZ^l of degree n, and in this case

$$1 + \left[\frac{n+4}{4}\right] + \left[\frac{n+1}{4}\right] = \left[\frac{n+2}{2}\right].$$

This completes the proof of the lemma. □

To complete the proof of the theorem, consider the homomorphism of graded rings

$$\mathbb{Z}[W, X, Y, Z]/\mathcal{I} \longrightarrow \mathcal{P}_*$$

induced by P. By Lemma 2 this is surjective. Now $\mathcal{P}_n$ is free of rank $[(n+2)/2]$ by Lemma 1, and the degree n part of $\mathbb{Z}[W, X, Y, Z]/\mathcal{I}$, which surjects to $\mathcal{P}_n$, is generated as a $\mathbb{Z}$-module by $[(n+2)/2]$ elements, according to Lemma 3. This is only possible if the degree n part of $\mathbb{Z}[W, X, Y, Z]/\mathcal{I}$ is also free, and the surjection is injective, and, therefore, an isomorphism. □

Remark 1. The generators W, X, Y and Z satisfy the following:

$$\begin{aligned} P(W) &= P_{t,z}(S^1), \\ P(X) &= P_{t,z}(S^1 \times S^1) - P_{t,z}(S^2), \\ P(Y) &= P_{t,z}(S^1 \times S^2) - P_{t,z}(S^3), \\ P(Z) &= P_{t,z}(S^2 \times S^2) - P_{t,z}(\mathbb{C}P^2). \end{aligned}$$

This shows that the definition of the Poincaré ring used in this section gives the same ring as the one defined in the introduction. Indeed, all elements of $\mathcal{P}_*$ as defined here are $\mathbb{Z}$-linear

combinations of Poincaré polynomials of closed orientable manifolds, and one can take S^1, S^2, S^3 and $\mathbb{C}P^2$ as generators. The generators W, X, Y and Z have the advantage of giving a simpler form for the relations generating the ideal $\mathcal{I}$.

Theorem 5 has the following immediate implication, showing that away from the prime 2 the oriented Poincaré ring is in fact a polynomial ring.

Corollary 1. *Let k be a field of characteristic $\neq 2$. Then $\mathcal{P}_* \otimes k$ is isomorphic to a polynomial ring $k[W, X]$ on two generators of degrees 1 and 2 respectively. For the generators one may take S^1 and S^2.*

Since products of S^1 and S^2 have vanishing Pontryagin numbers, Corollary 1 implies that there are no universal $\mathbb{Q}$-linear relations between Betti and Pontryagin numbers. This result also follows, in a less direct way, from [Kot10, Corollary 3]. The corresponding statement for congruences between integral linear combinations is slightly more subtle, and depends on the integral structure of the Poincaré ring.

Corollary 2. *Any non-trivial congruence between an integral linear combination of Betti numbers of oriented manifolds and an integral linear combination of Pontryagin numbers is a multiple of the mod 2 congruence between the Euler characteristic and the signature.*

Here, as always, the Betti numbers are considered modulo the symmetry induced by Poincaré duality. Non-trivial congruences are those in which the two sides do not vanish separately.

Proof. A linear combination of Betti numbers of oriented n-manifolds that is congruent mod m to a linear combination of Pontryagin numbers corresponds to a homomorphism $\varphi\colon \mathcal{P}_n \longrightarrow \mathbb{Z}_m$ that vanishes on all manifolds with zero Pontryagin numbers. Consider the generating elements W, X, Y and Z of $\mathcal{P}_*$ in Theorem 5. In terms of these elements, the 4-sphere satisfies

$$S^4 = W^4 - 4WY + 2Z \in \mathcal{P}_*.$$

Since any product with W, X, Y or S^4 as a factor has vanishing Pontryagin numbers, Theorem 5 together with this relation implies that the homomorphism φ descends to the degree n part of the quotient $\mathbb{Z}[Z]/2Z$. Now the mod 2 reduction of the Euler characteristic induces an isomorphism between $\mathbb{Z}[Z]/2Z$ and $\mathbb{Z}_2[z^4]$. Furthermore, the Euler characteristic is congruent mod 2 to the signature, which is a linear combination of Pontryagin numbers by the work of Thom. This completes the proof. □

Remark 2. Proceeding as above, one can define the unoriented Poincaré ring using $\mathbb{Z}_2$-Poincaré polynomials of manifolds that are not necessarily orientable. It is easy to see that this ring is a polynomial ring over $\mathbb{Z}$, isomorphic to $\mathbb{Z}[\mathbb{R}P^1, \mathbb{R}P^2]$.

3. The Hodge ring

To every closed Kähler manifold of complex dimension n we associate its Hodge polynomial

$$H_{x,y,z}(M) = \left(\sum_{p,q=0}^{n} h^{p,q}(M) \cdot x^p y^q \right) \cdot z^n \in \mathbb{Z}[x, y, z],$$

where the $h^{p,q}(M)$ are the Hodge numbers satisfying the Kähler constraints $h^{q,p} = h^{p,q} = h^{n-p,n-q}$. Like with the Poincaré polynomial, we have augmented the Hodge polynomial by the introduction of the additional variable z, which ensures that the Hodge polynomial defines an

embedding of the Hodge ring $\mathcal{H}_*$ defined in the introduction into the polynomial ring $\mathbb{Z}[x, y, z]$. This embedding preserves the grading if we set $\deg(x) = \deg(y) = 0$ and $\deg(z) = 1$.

The Hodge polynomial refines the Poincaré polynomial in the sense that if one sets $x = y = t$ and collects terms, the Hodge polynomial reduces to the Poincaré polynomial. (At the same time one has to replace z by z^2 since the real dimension of a Kähler manifold is twice its complex dimension.)

Unlike in the definition used in the introduction, we now define $\mathcal{H}_n$ to be the $\mathbb{Z}$-module of *all* polynomials

$$H_{x,y,z} = \left(\sum_{p,q=0}^{n} h^{p,q} \cdot x^p y^q \right) \cdot z^n \in \mathbb{Z}[x, y, z]$$

satisfying the constraints $h^{q,p} = h^{p,q} = h^{n-p,n-q}$. We will prove in Corollary 3 below that all elements of $\mathcal{H}_n$ are in fact $\mathbb{Z}$-linear combinations of Hodge polynomials of compact Kähler manifolds of complex dimension n, so that this definition agrees with the one in the introduction.

Lemma 4. *The $\mathbb{Z}$-module $\mathcal{H}_n$ is free of rank $[(n+2)/2] \cdot [(n+3)/2]$.*

Proof. Given the constraints $h^{q,p} = h^{p,q} = h^{n-p,n-q}$, visualized in the Hodge diamond, it is straightforward to write down a module basis for $\mathcal{H}_n$ with $[(n+2)/2] \cdot [(n+3)/2]$ elements. □

We define the Hodge ring by

$$\mathcal{H}_* = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \subset \mathbb{Z}[x, y, z].$$

This is a commutative ring with a grading given by the degree. (Recall that the degrees or weights of x, y and z are 0, 0 and 1 respectively.) Multiplication corresponds to taking the Cartesian product of Kähler manifolds, and the grading corresponds to the complex dimension. Its structure is completely described by the following theorem.

Theorem 6. *Let A and B have degree 1 and C have degree 2. The homomorphism*

$$H \colon \mathbb{Z}[A, B, C] \longrightarrow \mathcal{H}_*$$

given by

$$H(A) = (1 + xy) \cdot z, \quad H(B) = (x + y) \cdot z, \quad H(C) = xy \cdot z^2$$

is an isomorphism of graded rings.

This result can be proved by an argument that parallels the one we used in the proof of Theorem 5. We give a different proof, that illustrates a somewhat different point of view.

Proof. In order to prove the injectivity of H, we need to show that there is no non-trivial polynomial in A, B and C which maps to zero under H. Since there is always a prime number p, such that the mod p reduction of such a polynomial is non-trivial, the injectivity of H follows from the following stronger statement.

Lemma 5. *Let p be a prime number. The mod p reduction of the map H*

$$\tilde{H} \colon \mathbb{Z}_p[A, B, C] \longrightarrow \mathbb{Z}_p[x, y, z],$$

given by sending A, B and C to the mod p reductions of $H(A)$, $H(B)$ and $H(C)$, is injective.

Proof. Suppose the contrary and let n be the smallest degree in which $\tilde{H}$ is not injective. Then $\ker(\tilde{H})$ contains a non-trivial element of the form $C \cdot Q(A, B, C) + R(A, B)$, where $Q(A, B, C)$

and $R(A,B)$ are homogeneous polynomials with coefficients in $\mathbb{Z}_p$ of degrees $n-2$ and n respectively. If we set $y=0$, we obtain $R(z,xz)=0$ in $\mathbb{Z}_p[x,z]$. Since z and xz are algebraically independent in $\mathbb{Z}_p[x,z]$, we conclude that the polynomial R vanishes identically. Therefore, $C\cdot Q(A,B,C)\in\ker(\tilde{H})$. Since $\mathbb{Z}_p[x,y,z]$ is an integral domain in which $\tilde{H}(C)=xy\cdot z^2$ is a non-trivial element, we conclude that $Q(A,B,C)$ also lies in the kernel of $\tilde{H}$. This contradicts the minimality of n. □

It remains to prove the surjectivity of H. Counting the monomials in A, B, and C of degree n shows that the degree n part of the graded polynomial ring $\mathbb{Z}[A,B,C]$ is a free $\mathbb{Z}$-module of rank $N=[(n+2)/2]\cdot[(n+3)/2]$. By the injectivity of H, this is mapped isomorphically onto a submodule of $\mathcal{H}_n$, which by Lemma 4 is also a free $\mathbb{Z}$-module of rank N. Therefore, there are a basis $h_1,\dots,h_N$ of $\mathcal{H}_n$ and non-zero integers $a_1,\dots,a_N$ such that $a_1h_1,\dots,a_Nh_N$ is a basis of $\mathrm{Im}(H)$. It remains to show that the integers a_i are all equal to ±1. Suppose the contrary and let p be a prime number which divides a_i. Since $a_ih_i\in\mathrm{Im}(H)$, this is the image of a polynomial $S(A,B,C)$. The mod p reduction of S must be non-trivial, since otherwise a_ih_i/p would lie in the image of H. However, the mod p reduction of a_ih_i vanishes by assumption, which is a contradiction with Lemma 5. This completes the proof of the theorem. □

From now on we use the isomorphism H to identify A, B and C with their images in $\mathcal{H}_*$. The following corollary paraphrases Theorem 1 stated in the introduction, and explains that instead of A, B and C one may choose different generators for $\mathcal{H}_*$. Before we state it, note that by the Hodge index theorem the signature of manifolds induces a ring homomorphism $\sigma\colon\mathcal{H}_*\longrightarrow\mathbb{Z}[z]$, given by $x\mapsto-1$, $y\mapsto1$.

Corollary 3. *Let E be an elliptic curve, L the projective line and let S be an element in $\mathcal{H}_2$ with signature ±1. (For instance, S might be a Kähler surface with signature ±1.) Then, $\mathcal{H}_*$ is isomorphic to the polynomial ring $\mathbb{Z}[E,L,S]$.*

Proof. First of all, note the identities $A=L$ and $B=E-L$, which allow us to replace the generators A and B in degree 1 by E and L. We may represent the element S with respect to the basis A^2, AB, B^2 and C of $\mathcal{H}_2$, given by Theorem 6. It remains to show that, in this representation, the basis element C occurs with coefficient ±1. Since A and B have zero signature and C has signature -1, this is equivalent to S having signature ±1, which is true by assumption. □

Remark 3. We have now proved that all formal Hodge polynomials are indeed $\mathbb{Z}$-linear combinations of Hodge polynomials of Kähler manifolds. This shows that the definition of $\mathcal{H}_*$ given at the beginning of this section gives the same ring as the definition in the introduction, and it proves statement (3) from the introduction.

The last corollary also leads to the following result, which generalizes [Kot09, Theorem 6], proved there rather indirectly.

Corollary 4. *The mod m reduction of a $\mathbb{Z}$-linear combination of Hodge numbers equals the mod m reduction of a linear combination of Pontryagin numbers if and only if, modulo m, it is a multiple of the signature.*

Proof. If in complex dimension $2n$ a $\mathbb{Z}$-linear combination of Pontryagin numbers equals a linear combination of Hodge numbers, then it can be considered as a homomorphism φ on $\mathcal{H}_{2n}$. The domain is spanned by products of E, L and S, but any product with a complex curve as a factor has trivial Pontryagin numbers. Thus φ factors through the projection $\mathbb{Z}[L,E,S]\longrightarrow\mathbb{Z}[S]$, which

we can identify with the signature homomorphism, since the signature of S is ± 1. Conversely, the signature is a linear combination of Pontryagin numbers by the classical results of Thom. □

Returning to the generators A, B and C for $\mathcal{H}_*$ we can prove the following result, which implies Theorem 2 stated in the introduction.

Theorem 7. *Let $\mathcal{I} \subset \mathcal{H}_*$ be the ideal generated by differences of birational smooth complex projective varieties. Then $\mathcal{I} = (C) = \ker(b)$, where $C = xy \cdot z^2$ and $b\colon \mathcal{H}_* \longrightarrow \mathbb{Z}[y,z]$ is given by setting $x = 0$ in the Hodge polynomials.*

Proof. If S is a Kähler surface and $\hat{S}$ its blowup at a point, then $\hat{S} - S = C$, and so $(C) \subset \mathcal{I}$.

The homomorphism b sends the Hodge polynomial in degree n to $(h^{0,0} + h^{0,1}y + \cdots + h^{0,n}y^n)z^n$. As the $h^{0,q}$ are birational invariants, cf. [GH78, p. 494], we have $\mathcal{I} \subset \ker(b)$. From the proof of Theorem 6 we know already that there are no universal relations between the Hodge numbers, other than the ones generated by the Kähler symmetries, and so the image of b in degree n is a free $\mathbb{Z}$-module of rank $n+1$. Since $(C) \subset \ker(b)$, this means that b maps $\mathbb{Z}[A,B]$ isomorphically onto $\mathrm{Im}(b)$, and so $(C) = \ker(b)$. □

This theorem tells us exactly which linear combinations of Hodge numbers are birational invariants of projective varieties, or of compact Kähler manifolds. Indeed, any homomorphism $\varphi\colon \mathcal{H}_n \longrightarrow M$ of $\mathbb{Z}$-modules that vanishes on $\mathcal{I} \cap \mathcal{H}_n$ factors through b. This proves Theorem 2 stated in the introduction.

We already mentioned the homomorphism $\sigma\colon \mathcal{H}_* \longrightarrow \mathbb{Z}[z]$ given by the signature. It is a specialization (for $y = 1$) of the Hirzebruch genus

$$\chi\colon \mathcal{H}_* \longrightarrow \mathbb{Z}[y,z]$$

defined by setting $x = -1$ in the Hodge polynomials. Consider a polynomial

$$(\chi_0 + \chi_1 y + \cdots + \chi_n y^n) \cdot z^n \in \mathrm{Im}(\chi).$$

By Serre duality in $\mathcal{H}_n$, this must satisfy the constraint $\chi_p = (-1)^n \chi_{n-p}$. Let $\mathcal{H}ir_n$ be the $\mathbb{Z}$-module of all polynomials of the form $(\chi_0 + \chi_1 y + \cdots + \chi_n y^n)z^n \in \mathbb{Z}[y,z]$ satisfying this constraint. It is clear that this is a free $\mathbb{Z}$-module of rank $[(n+2)/2]$, and that

$$\mathcal{H}ir_* = \bigoplus_{n=0}^{\infty} \mathcal{H}ir_n \subset \mathbb{Z}[y,z]$$

is a graded commutative ring.

Theorem 8. *The Hirzebruch genus defines a surjective homomorphism $\chi\colon \mathcal{H}_* \longrightarrow \mathcal{H}ir_*$ of graded rings, whose kernel is the principal ideal in $\mathcal{H}_*$ generated by an elliptic curve. In particular $\mathcal{H}ir_*$ is a polynomial ring over $\mathbb{Z}$ with one generator in degree 1 and one in degree 2. As generators one may choose $\mathbb{C}P^1$ and $\mathbb{C}P^2$.*

Proof. It is clear that χ is a homomorphism of graded rings, and that elliptic curves are in its kernel. Identifying $\mathcal{H}_*$ with $\mathbb{Z}[E, \mathbb{C}P^1, \mathbb{C}P^2]$, the Hirzebruch genus factors through the projection $\mathbb{Z}[E, \mathbb{C}P^1, \mathbb{C}P^2] \longrightarrow \mathbb{Z}[\mathbb{C}P^1, \mathbb{C}P^2]$, and we have to show that the induced homomorphism $\mathbb{Z}[\mathbb{C}P^1, \mathbb{C}P^2] \longrightarrow \mathcal{H}ir_*$ is an isomorphism. This follows from the proof of Theorem 6, where we showed that there are no unexpected relations between the Hodge numbers. In particular, there are no non-trivial relations between the coefficients $\chi_0, \chi_1, \ldots, \chi_{[n/2]}$. Alternatively one can show that $\mathbb{Z}[\mathbb{C}P^1, \mathbb{C}P^2] \longrightarrow \mathcal{H}ir_*$ is an isomorphism by elementary manipulations using $\chi(\mathbb{C}P^1) = (1-y)z$ and $\chi(\mathbb{C}P^2) = (1 - y + y^2)z^2$. □

Remark 4. With coefficients in $\mathbb{Q}$, it is well known that the image of the Hirzebruch genus is a polynomial ring on the images of $\mathbb{C}P^1$ and $\mathbb{C}P^2$. That this also holds over $\mathbb{Z}$ was recently made explicit in [Sch, Remark 7.6]. There, as everywhere in the literature, the Hirzebruch genus is identified with the Todd genus on the complex bordism ring using the Hirzebruch–Riemann–Roch theorem. However, by its very definition, it should be considered on the Hodge ring instead, which is a much simpler object than the bordism ring, and in particular is finitely generated. By HRR, the two interpretations give the same image, since the bordism ring is generated, over $\mathbb{Z}$, by Kähler manifolds, compare § 6.1 below.

Theorem 8 tells us that there are no universal relations between the Hodge and Chern numbers other than the Hirzebruch–Riemann–Roch relations.

Corollary 5. *The mod m reduction of a $\mathbb{Z}$-linear combination of Hodge numbers of smooth complex projective varieties equals a linear combination of Chern numbers if and only if, mod m and modulo Kähler symmetries, it is a linear combination of the χ_p.*

Proof. Since products with an elliptic curve as a factor have trivial Chern numbers, any linear combination of Hodge numbers that equals a combination of Chern numbers must factor through the projection $\mathcal{H}_* \longrightarrow \mathcal{H}_*/(E)$. By Theorem 8, this projection is the Hirzebruch genus χ. Conversely, by the Hirzebruch–Riemann–Roch theorem, the coefficients of χ are expressed as linear combinations of Chern numbers via the Todd polynomials. □

4. The comparison map and the Poincaré ring of Kähler manifolds

In this section we analyse the comparison map

$$f\colon \mathcal{H}_* \longrightarrow \mathcal{P}_*$$
$$x \longmapsto t, \quad y \longmapsto t, \quad z \longmapsto z^2$$

given by forgetting the Kähler structure on elements of $\mathcal{H}_*$, thus specializing Hodge polynomials to Poincaré polynomials. This map doubles the degree, since the real dimension of a Kähler manifold is twice its complex dimension. Here are the main properties of this homomorphism.

Proposition 1. (1) *The image of f consists of all elements of $\mathcal{P}_* \subset \mathbb{Z}[t,z]$ of even degree, whose coefficients of odd powers of t are even.*

(2) *The kernel of f is a principal ideal in $\mathcal{H}_*$ generated by the following homogeneous element G of degree 2:*

$$G = 4\mathbb{C}P^2 - 3L^2 + E^2 - 2EL.$$

Proof. In § 3 we defined $\mathcal{H}_*$ to be generated by all formal Hodge polynomials

$$\left(\sum_{p,q=0}^{n} h^{p,q}\cdot x^p y^q\right)\cdot z^n,$$

in $\mathbb{Z}[x,y,z]$, satisfying the Kähler symmetries $h^{p,q} = h^{q,p} = h^{n-p,n-q}$. Serre duality $h^{p,q} = h^{n-p,n-q}$ implies Poincaré duality for the image under f, whereas the symmetry $h^{p,q} = h^{q,p}$ implies that the image has even odd-degree Betti numbers. Finally, since f doubles the degree, its image is concentrated in even degrees. Conversely, it is straightforward to check that the elements e_n^{2n}, e_k^{2n} with even $k<n$, and $2e_k^{2n}$ with odd $k<n$ of $\mathcal{P}_{2n}$ in Lemma 1 are images of formal Hodge polynomials. This establishes the first part of the proposition.

For the second part, we note that $G = 4\mathbb{C}P^2 - 3L^2 + E^2 - 2EL$ has zero Betti numbers and therefore lies in the kernel of f. Thus f induces a homomorphism $\hat{f}\colon \mathcal{H}_*/(G) \longrightarrow \mathcal{P}_*$. By the first part of the proposition proved above, the image of f, equivalently $\hat{f}$, in degree $2n$ is a free $\mathbb{Z}$-module of rank $n+1$. By Corollary 3, the degree n part of $\mathcal{H}_*/(G)$ is generated as a $\mathbb{Z}$-module by $n+1$ elements. Therefore, $\hat{f}$ is injective, and an isomorphism onto $\mathrm{Im}(f) \subset \mathcal{P}_*$. □

By the first part of this proposition, a basis for $\mathrm{Hom}(f(\mathcal{H}_n), \mathbb{Z})$ is given by the even-degree Betti numbers and the halves of the odd-degree Betti numbers, both up to the middle dimension only because of Poincaré duality. In particular, the only non-trivial congruences satisfied by the Betti numbers of Kähler manifolds are the vanishing mod 2 of the odd-degree Betti numbers.

Proceeding as in the definition of the Poincaré ring $\mathcal{P}_*$ of oriented manifolds in § 2, we define the Poincaré ring of Kähler manifolds. This ring is the image of the comparison map f in $\mathcal{P}_*$. Thus, Proposition 1 yields the following theorem.

Theorem 9. *The Poincaré ring of Kähler manifolds* $\mathrm{Im}(f)$ *is isomorphic to*

$$\mathbb{Z}[L, E, \mathbb{C}P^2]/(4\mathbb{C}P^2 - 3L^2 + E^2 - 2EL),$$

where $L = \mathbb{C}P^1$ *is the projective line and* E *an elliptic curve.*

Using this theorem, we can determine all universal relations between Betti and Pontryagin numbers of Kähler manifolds. Since in odd complex dimensions there are no non-trivial Pontryagin numbers, we can restrict ourselves to even complex dimensions. In these dimensions, for Kähler manifolds only, Corollary 2 is strengthened as follows.

Corollary 6. *Any non-trivial congruence between an integral linear combination of Betti numbers of Kähler manifolds of even complex dimension* $2n$ *and an integral linear combination of Pontryagin numbers is a multiple of the following congruence between the Euler characteristic and the signature:*

$$e \equiv (-1)^n \sigma \mod 4. \tag{1}$$

The word non-trivial in the formulation is meant to indicate that we ignore congruences where both sides vanish separately. This is necessary because the odd-degree Betti numbers are all even.

Proof. The signature is a linear combination of Pontryagin numbers by the work of Thom. That it satisfies the congruence (1) for compact Kähler manifolds follows from the Hodge index theorem.

Conversely, suppose we have a $\mathbb{Z}$-linear combination of Betti numbers that, on all Kähler manifolds of complex dimension $2n$, is congruent to a linear combination of Pontryagin numbers modulo m, but does not vanish identically mod m. Such a linear combination corresponds to a homomorphism φ from the degree $4n$ part of the Poincaré ring of Kähler manifolds to $\mathbb{Z}_m$ that vanishes on all elements with zero Pontryagin numbers. Since the Pontryagin numbers vanish on manifolds that are products with a complex curve as a factor, Theorem 9 shows that φ factors through the degree $4n$ part of $\mathbb{Z}[\mathbb{C}P^2]/(4\mathbb{C}P^2)$. Now the mod 4 reduction of the Euler characteristic gives an isomorphism between $\mathbb{Z}[\mathbb{C}P^2]/(4\mathbb{C}P^2)$ and $\mathbb{Z}_4[z^4]$. This completes the proof. □

Replacing the Pontryagin numbers by the Chern numbers of Kähler manifolds, we obtain the following corollary.

Corollary 7. *A $\mathbb{Z}$-linear combination of Betti numbers of Kähler manifolds is congruent mod m to a non-trivial linear combination of Chern numbers if and only if, mod m, it is a multiple of the Euler characteristic.*

Again we do not consider congruences where the two sides vanish separately.

Proof. Since the Euler characteristic of a Kähler manifold equals the top Chern number c_n, one direction is clear. For the converse, assume that, in complex dimension n, the mod m reduction of some $\mathbb{Z}$-linear combination of Chern numbers equals a linear combination of Betti numbers. This corresponds to a non-trivial homomorphism from the degree $2n$ part of the Poincaré ring of Kähler manifolds to $\mathbb{Z}_m$. Since any product with an elliptic curve has trivial Chern numbers, Theorem 9 shows that this homomorphism descends to a homomorphism from the degree $2n$ part of $\mathbb{Z}[L, \mathbb{C}P^2]/(4\mathbb{C}P^2 - 3L^2)$ to $\mathbb{Z}_m$. Upon identifying this ring with the subring of $\mathbb{Z}[z^2]$ generated by $2z^2$ and z^4, the projection from the Poincaré ring of Kähler manifolds to $\mathbb{Z}[L, \mathbb{C}P^2]/(4\mathbb{C}P^2 - 3L^2)$ is identified with the Euler characteristic, obtained by setting $t = -1$ in the Poincaré polynomials. This completes the proof. □

5. The Hirzebruch problem for Hodge numbers

In this section we solve Hirzebruch's problem concerning Hodge numbers by proving Theorem 3 stated in the introduction. The following is the first step in its proof.

Theorem 10. *The ideal in the Hodge ring $\mathcal{H}_*$ generated by the differences of homeomorphic smooth complex projective varieties coincides with the kernel of the forgetful map $f\colon \mathcal{H}_* \longrightarrow \mathcal{P}_*$.*

Proof. Let $\mathcal{I} \subset \mathcal{H}_*$ be the ideal generated by

$$\{M - N \mid M, N \text{ homeomorphic projective varieties of dimension } n\},$$

for all n. These are differences of smooth complex projective varieties of complex dimension n that are homeomorphic, without any assumption about compatibility of their orientations under homeomorphisms.

Since Poincaré polynomials are homeomorphism invariants, it is clear that $\mathcal{I} \subset \ker(f)$. To prove $\ker(f) \subset \mathcal{I}$ we use Proposition 1, telling us that $\ker(f)$ is a principal ideal generated by an element G in degree 2. This G has the property that all its Betti numbers vanish, and its signature equals $+4$. We only have to prove that $G \in \mathcal{I}$.

By the results of [Kot92] there are many pairs (X, Y) of simply connected projective surfaces of non-zero signature that are orientation-reversingly homeomorphic with respect to the orientations given by the complex structures. The only divisibility condition that has to be satisfied in all cases is that the signatures must be even. More specifically, by [Kot92, Theorem 3.7], we can choose two such pairs (X_1, Y_1) and (X_2, Y_2) with the property that the greatest common divisor of the signatures $\sigma(X_1)$ and $\sigma(X_2)$ is 2. Then there are integers a and b such that

$$a\sigma(X_1) + b\sigma(X_2) = 2. \tag{2}$$

We now claim that the following identity holds:

$$H_{x,y,z}(G) = a(H_{x,y,z}(X_1) - H_{x,y,z}(Y_1)) + b(H_{x,y,z}(X_2) - H_{x,y,z}(Y_2)). \tag{3}$$

Since $X_i - Y_i \in \mathcal{I}$, this proves that $G \in \mathcal{I}$.

To prove (3) note that the Betti numbers vanish on both the left-hand and the right-hand sides. Therefore, to check that all Hodge numbers agree, we only have to check the equality of

the signatures, as follows:

$$\sigma(a(X_1 - Y_1) + b(X_2 - Y_2)) = 2\sigma(aX_1 + bX_2) = 4 = \sigma(G),$$

where the first equality comes from the fact that X_i and Y_i are orientation-reversingly homeomorphic and the second equality comes from (2). This completes the proof of the theorem. □

Next we consider differences of diffeomorphic, not just homeomorphic, projective varieties.

Theorem 11. *In degrees $n \geqslant 3$ the kernel of $f\colon \mathcal{H}_n \longrightarrow \mathcal{P}_{2n}$ is generated as a $\mathbb{Z}$-module by differences of diffeomorphic smooth complex projective varieties.*

In all degrees the intersection $\ker(f) \cap \ker(\sigma)$ is generated as a $\mathbb{Z}$-module by differences of smooth complex projective varieties that are orientation-preservingly diffeomorphic with respect to the orientations induced by the complex structures.

Proof. By the proof of Theorem 10, the ideal $\ker(f)$ is generated by differences of pairs of homeomorphic simply connected algebraic surfaces (X_i, Y_i). Identifying $\mathcal{H}_*$ with $\mathbb{Z}[E, \mathbb{C}P^1, \mathbb{C}P^2]$, we see that the kernel of $f\colon \mathcal{H}_n \longrightarrow \mathcal{P}_{2n}$ is generated as a $\mathbb{Z}$-module by products of the $X_i - Y_i$ with E, $\mathbb{C}P^1$ and $\mathbb{C}P^2$.

By a result of Wall [Wal64], the smooth four-manifolds X_i and Y_i are smoothly h-cobordant. It follows that $X_i \times \mathbb{C}P^j$ and $Y_i \times \mathbb{C}P^j$ are also h-cobordant, and are therefore diffeomorphic by Smale's h-cobordism theorem [Sma62]. Products of $X_i - Y_i$ with powers of E, therefore not involving a $\mathbb{C}P^j$, are handled by the following Lemma, which is a well-known consequence of the s-cobordism theorem of Barden, Mazur and Stallings; see [Ker65, p. 41/42].

Lemma 6. *Let M and N be h-cobordant manifolds of dimension $\geqslant 5$. Then $M \times S^1$ and $N \times S^1$ are diffeomorphic.*

This shows that the products $X_i \times E$ and $Y_i \times E$ are diffeomorphic, completing the proof of the first statement.

For the second statement note that $\ker(f) \cap \ker(\sigma)$ vanishes in degrees < 3. Therefore, we only have to consider the degrees already considered in the first part. The generators considered there all have zero signature, except the products of $X_i - Y_i$ with pure powers of $\mathbb{C}P^2$. This implies that the products of $X_i - Y_i$ with monomials in E, $\mathbb{C}P^1$ and $\mathbb{C}P^2$ that involve at least one of the curves generate $\ker(f) \cap \ker(\sigma)$. Since E and $\mathbb{C}P^1$ admit orientation-reversing self-diffeomorphisms, it follows that $X_i \times E$ and $Y_i \times E$, respectively $X_i \times \mathbb{C}P^1$ and $Y_i \times \mathbb{C}P^1$, are not just diffeomorphic, as proved above, but that the diffeomorphism may be chosen to preserve the orientations. This completes the proof. □

We can now give a complete answer to Hirzebruch's question concerning Hodge numbers.

Proof of Theorem 3. We consider integral linear combinations of Hodge numbers as homomorphisms $\varphi\colon \mathcal{H}_n \longrightarrow \mathbb{Z}_m$. If a linear combination of Hodge numbers defines an unoriented homeomorphism invariant, then by Theorem 10 the corresponding homomorphism φ factors through f. Looking at the description of $\mathrm{Im}(f)$ in Proposition 1, we see that every homeomorphism-invariant linear combination of Hodge numbers is a combination of the even-degree Betti numbers and the halves of the odd-degree Betti numbers. By the first part of Theorem 11, the same conclusion holds for unoriented diffeomorphism invariants in dimensions $n \neq 2$.

Combining the above discussion with the second part of Theorem 11 completes the proof of Theorem 3. □

Example 1. By the results of [Kot92] used above, the signature itself is not a homeomorphism invariant of smooth complex projective varieties. However, the reduction mod 4 of the signature is a homeomorphism invariant, since by the proof of Theorem 10, it vanishes on the ideal $\mathcal{I} = \ker(f)$. Theorem 3 then tells us that the signature of a Kähler manifold is congruent mod 4 to a linear combination of even-degree Betti numbers and halves of odd-degree Betti numbers. This latter fact also follows from the Hodge index theorem, which gives the precise congruence (1).

6. The Chern–Hodge ring

6.1 Unitary bordism

We now recall the classical results about the complex bordism ring $\Omega^U_* = \bigoplus_{n=0}^{\infty} \Omega^U_n$ that we shall need. By results of Milnor [Mil60, Tho95] and Novikov [Nov62] this is a polynomial ring over $\mathbb{Z}$ on countably many generators β_i, one for every complex dimension i. In particular, the degree n part Ω^U_n is a free $\mathbb{Z}$-module of rank $p(n)$, the number of partitions of n. Two stably almost complex manifolds of the same dimension have the same Chern numbers if and only if they represent the same element in Ω^U_*.

The β_i are commonly referred to as a basis sequence, and we will need to discuss some special choices of such basis sequences. An element $\beta_n \in \Omega^U_n$ can be taken as a generator over $\mathbb{Z}$ if and only if a certain linear combination of Chern numbers s_n, referred to as the Thom–Milnor number, satisfies $s_n(\beta_n) = \pm 1$ if $n+1$ is not a prime power, and $s_n(\beta_n) = \pm p$ if $n+1$ is a power of the prime p.

In the case of $\Omega^U_* \otimes \mathbb{Q}$ one may take $\beta_i = \mathbb{C}P^i$ as a basis sequence, but this is not a basis sequence over $\mathbb{Z}$. Milnor proved that one can obtain a basis sequence over $\mathbb{Z}$ by considering formal $\mathbb{Z}$-linear combinations of complex projective spaces and of smooth hypersurfaces $H \subset \mathbb{C}P^k \times \mathbb{C}P^{i+1-k}$ of bidegree $(1,1)$, cf. [Tho95] and [Mil07, pp. 249–252]. It follows that one may take (disconnected) projective, in particular Kähler, manifolds for the generators of Ω^U_* over $\mathbb{Z}$. These projective manifolds are very special, in that they are birational to $\mathbb{C}P^i$.

Lemma 7. *Milnor manifolds, that is, smooth hypersurfaces $H \subset \mathbb{C}P^k \times \mathbb{C}P^{i+1-k}$ of bidegree $(1,1)$, are rational.*

Proof. Let x and y be homogeneous coordinates on $\mathbb{C}P^k$ respectively $\mathbb{C}P^{i+1-k}$. In an affine chart $\mathbb{C}^k \times \mathbb{C}^{i+1-k} = \mathbb{C}^{i+1}$ given by $x_0 \neq 0 \neq y_0$, say, the defining equation of H of bidegree $(1,1)$ in x and y becomes a quadratic equation in the coordinates of $\mathbb{C}^{i+1}$. Therefore H is birational to an irreducible quadric in $\mathbb{C}P^{i+1}$, which is well known to be rational. □

Finally the Todd genus $\mathrm{Td}\colon \Omega^U_* \longrightarrow \mathcal{H}ir_*$ is the ring homomorphism sending a bordism class $[M]$ to $(\mathrm{Td}_0(M) + \mathrm{Td}_1(M)y + \cdots + \mathrm{Td}_n(M)y^n)z^n$, where the Td_p are certain combinations of Chern numbers. By the Hirzebruch–Riemann–Roch theorem one has $\mathrm{Td}_p = \chi_p = \sum_q (-1)^q h^{p,q}$.

6.2 Combining the Hodge and bordism rings

We now consider finite linear combinations of equidimensional compact Kähler manifolds with coefficients in $\mathbb{Z}$, and identify two such linear combinations if they have the same dimensions and the same Hodge and Chern numbers. The set of equivalence classes is naturally a graded ring, graded by the dimension, with multiplication induced by the Cartesian product of Kähler manifolds. We call this the Chern–Hodge ring $\mathcal{CH}_*$.

The degree n part $\mathcal{CH}_n$ of the Chern–Hodge ring is the diagonal submodule $\Delta_n \subset \mathcal{H}_n \oplus \Omega^U_n$ generated by all

$$(H_{x,y,z}(M^n), [M^n]) \in \mathcal{H}_n \oplus \Omega^U_n,$$

where M runs over compact Kähler manifolds of complex dimension n and the square brackets denote bordism classes.

Proposition 2. *The diagonal submodule Δ_n is the kernel of the surjective homomorphism*

$$h\colon \mathcal{H}_n \oplus \Omega_n^U \longrightarrow \mathcal{H}ir_n$$
$$(H_{x,y,z}(M), [N]) \longmapsto \chi(M) - \mathrm{Td}(N),$$

where $\chi\colon \mathcal{H}_* \longrightarrow \mathcal{H}ir_*$ is the Hirzebruch genus, and $\mathrm{Td}\colon \Omega_*^U \longrightarrow \mathcal{H}ir_*$ is the Todd genus.

Proof. The surjectivity of h follows from the surjectivity of χ proved in Theorem 8.

By the Hirzebruch–Riemann–Roch theorem $\Delta_n \subset \ker(h)$. To check the reverse inclusion consider an element $(H_{x,y,z}(M), [N]) \in \ker(h)$. This means $\chi(M) = \mathrm{Td}(N)$, and so, applying HRR to N, $\chi(M) = \chi(N)$. Since by Theorem 8 the kernel of χ is the principal ideal generated by an elliptic curve E, we conclude that in the Hodge ring the difference of M and N is of the form $E \cdot P$, where P is a homogeneous polynomial of degree $n-1$ in the generators of $\mathcal{H}_*$. Thus in $\mathcal{H}_n \oplus \Omega_n^U$ we may write

$$(H_{x,y,z}(M), [N]) = (H_{x,y,z}(N), [N]) + (H_{x,y,z}(E \cdot P), 0).$$

Since an elliptic curve E represents zero in the bordism ring, we have $(H_{x,y,z}(E \cdot P), 0) = (H_{x,y,z}(E \cdot P), [E \cdot P])$, and so the second summand on the right-hand side is in the diagonal submodule. As the first summand is trivially in Δ_n, we have now proved $\ker(h) \subset \Delta_n$. □

As a consequence of Proposition 2, $\mathcal{CH}_n = \Delta_n$ is a free $\mathbb{Z}$-module of rank

$$\begin{aligned}\operatorname{rk} \mathcal{CH}_n &= \operatorname{rk} \mathcal{H}_n + \operatorname{rk} \Omega_n^U - \operatorname{rk} \mathcal{H}ir_n \\ &= \left[\frac{n+2}{2}\right] \cdot \left[\frac{n+3}{2}\right] + p(n) - \left[\frac{n+2}{2}\right] \\ &= \left[\frac{n+2}{2}\right] \cdot \left[\frac{n+1}{2}\right] + p(n).\end{aligned}$$

The structure of the Chern–Hodge ring is described by the following result.

Theorem 12. *Let $\beta_1 = \mathbb{C}P^1, \beta_2, \beta_3, \ldots$ be $\mathbb{Z}$-linear combinations of Kähler manifolds forming a basis sequence for the complex bordism ring Ω_*^U, and let $P_i(E, \beta_1, \beta_2)$ be the unique polynomial in E, β_1 and β_2 having the same image in the Hodge ring as β_i. Then the Chern–Hodge ring $\mathcal{CH}_*$ is isomorphic as a graded ring to the quotient of $\mathbb{Z}[E, \beta_1, \beta_2, \beta_3, \ldots]$ by the ideal $\mathcal{I}$ generated by all $E(\beta_i - P_i(E, \beta_1, \beta_2))$.*

Proof. In degree 2 the Thom–Milnor number s_2 of a Kähler surface equals $c_1^2 - 2c_2$, which is 3 times the signature. Since β_2 is a generator of the bordism ring, we have $s_2(\beta_2) = \pm 3$, so β_2 has signature ± 1. By Corollary 3 this means that $\mathcal{H}_* = \mathbb{Z}[E, \beta_1, \beta_2]$. Therefore, for each β_i there is indeed a unique polynomial $P_i(E, \beta_1, \beta_2)$ having the same image as β_i in $\mathcal{H}_*$.

Consider the canonical ring homomorphism

$$\phi\colon \mathbb{Z}[E, \beta_1, \beta_2, \beta_3, \ldots] \longrightarrow \mathcal{CH}_*.$$

We first prove that ϕ is surjective. Let M be a compact Kähler manifold of dimension n, and $[M] \in \Omega_n^U$ its bordism class. We need to show that $(H_{x,y,z}(M), [M]) \in \mathrm{Im}(\phi)$. Since the β_i form a basis sequence for the bordism ring, there is a unique homogeneous polynomial P of degree n in the β_i such that $[M] = [P(\beta_i, \ldots, \beta_n)] \in \Omega_n^U$. We then have $\phi(P) = (H_{x,y,z}(P), [M]) \in \mathcal{CH}_n$. Moreover, $H_{x,y,z}(P) - H_{x,y,z}(M)$ is in the kernel of the Hirzebruch genus, which by Theorem 8

is the ideal $(E) \subset \mathcal{H}_*$. Thus, in $\mathcal{H}_*$ we may write $M = P + EQ$, where Q is a homogeneous polynomial of degree $n-1$ in E, β_1 and β_2. Since E maps to zero in the bordism ring, we conclude $\phi(P+EQ) = (H_{x,y,z}(M), [M])$. This completes the proof of surjectivity.

Finally we need to show that $\ker(\phi) = \mathcal{I}$. By the definition of $\mathcal{I}$, we have $\mathcal{I} \subset \ker(\phi)$, and so ϕ descends to the quotient $\mathbb{Z}[E, \beta_1, \beta_2, \beta_3, \ldots]/\mathcal{I}$. The degree n part of this quotient surjects to $\mathcal{CH}_n$, which is a free module of rank

$$\left[\frac{n+2}{2}\right] \cdot \left[\frac{n+1}{2}\right] + p(n),$$

where $p(n) = \operatorname{rk} \Omega_n^U$ is the number of partitions of n. Looking at the definition of $\mathcal{I}$ we see that the degree n part of the quotient $\mathbb{Z}[E, \beta_1, \beta_2, \beta_3, \ldots]/\mathcal{I}$ is generated as a $\mathbb{Z}$-module by $\operatorname{rk} \mathcal{CH}_n$ many monomials. Since we know already that ϕ is surjective, this shows that ϕ is injective, and therefore an isomorphism. □

We can now generalize Theorem 7 from the Hodge to the Chern–Hodge ring.

Theorem 13. *Let $\mathcal{I} \subset \mathcal{CH}_*$ be the ideal generated by differences of birational smooth complex projective varieties. Then there is a basis sequence for the bordism ring with $\beta_1 = \mathbb{C}P^1$ and $\beta_i \in \mathcal{I}$ for all $i \geqslant 2$. Furthermore, $\mathcal{I}$ is the kernel of the composition*

$$\mathcal{CH}_* \xrightarrow{p} \mathcal{H}_* \xrightarrow{b} \mathbb{Z}[y,z],$$

where $p\colon \mathcal{CH}_* \longrightarrow \mathcal{H}_*$ is the projection and $b\colon \mathcal{H}_* \longrightarrow \mathbb{Z}[y,z]$ is given by setting $x = 0$ in the Hodge polynomials.

Proof. Take $\beta_1 = \mathbb{C}P^1$, and $\mathbb{C}P^2 - \mathbb{C}P^1 \times \mathbb{C}P^1 = -C$ as the generator β_2 in degree 2. In higher degrees we take the Milnor generators, which are formal linear combinations of projective spaces and of Milnor manifolds, and, like in degree 2, subtract from each projective space or Milnor manifold a copy of $\beta_1^n = \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$. This does not change the property of being generators (over $\mathbb{Z}$), but, after this subtraction, we have generators β_i which for $i \geqslant 2$ are contained in $\mathcal{I}$ by Lemma 7. This completes the proof of the first statement. For the second statement note that by Theorem 7 the ideal $\mathcal{I}$ is contained in the kernel of $b \circ p$. Conversely, our choice of generators shows that $\ker(b \circ p) \subset \mathcal{I}$. □

As a consequence of this result, Theorem 2 holds for combinations of Hodge and Chern numbers.

Corollary 8. *The mod m reduction of an integral linear combination of Hodge and Chern numbers is a birational invariant of smooth complex projective varieties if and only if after adding a suitable combination of the $\chi_p - \mathrm{Td}_p$ it is congruent to a linear combination of the $h^{0,q}$ plus a linear combination of Chern numbers that vanishes mod m when evaluated on any smooth complex projective variety.*

One should keep in mind that the Hodge numbers in this statement are, as always, taken modulo the Kähler symmetries. The corresponding statement over $\mathbb{Q}$ follows from the statement about congruences.

7. The general Hirzebruch problem

Finally we address the general version of Hirzebruch's Problem 31 from [Hir54] asking which linear combinations of Hodge and Chern numbers are topological invariants. This combines the

work about Hodge numbers in § 5 above with the work on Chern numbers in [Kot12]. The first step is the following result.

Theorem 14. *The ideal $\mathcal{J}$ in $\mathcal{CH}_* \otimes \mathbb{Q}$ generated by differences of homeomorphic projective varieties is the kernel of the forgetful homomorphism*

$$F\colon \mathcal{CH}_* \otimes \mathbb{Q} \longrightarrow \mathcal{P}_* \otimes \mathbb{Q}.$$

In degrees $\geqslant 3$ this ideal coincides with the one generated by differences of diffeomorphic projective varieties.

Proof. Since Poincaré polynomials are homeomorphism invariants, it is clear that $\mathcal{J} \subset \ker(F)$.

By [Kot12, Theorem 10] there is a basis sequence $\beta_1 = \mathbb{C}P^1, \beta_2, \beta_3, \ldots$ for $\Omega^U_* \otimes \mathbb{Q}$ with $\beta_i \in \mathcal{J}$ for all $i \geqslant 2$. On the one hand, this means that, in the description of $\mathcal{CH}_* \otimes \mathbb{Q}$ as a quotient of the polynomial ring $\mathbb{Q}[E, \beta_1, \beta_2, \beta_3, \ldots]$ given by Theorem 12, the only monomials in the generators whose residue classes are not necessarily in $\mathcal{J}$ are those involving only E and β_1. On the other hand, it is clear from Corollary 1 that the residue class of a non-trivial polynomial in E and β_1 cannot be in $\ker(F)$. Thus $\ker(F)$ is the ideal generated by the β_i with $i \geqslant 2$, and is therefore contained in $\mathcal{J}$. This proves the first statement in the theorem.

For the second statement note that the β_i used above are in fact differences of diffeomorphic projective varieties as soon as $i \geqslant 3$, see [Kot12, Theorem 9], and that the same is true for $\beta_1 \cdot \beta_2$ and $\beta_2 \cdot \beta_2$. The generator β_2 is a difference of orientation-reversingly homeomorphic simply connected algebraic surfaces X and Y. As in the proof of Theorem 11 above it follows from Lemma 6 that $E \times X$ and $E \times Y$ are diffeomorphic, and so $E \cdot \beta_2$ is also a difference of diffeomorphic projective varieties. □

Next we look at oriented topological invariants. For this it is convenient to introduce the oriented analogue of the Chern–Hodge ring. Consider formal $\mathbb{Z}$-linear combinations of equidimensional closed oriented smooth manifolds, and identify two such combinations if they have the same dimension, the same Betti numbers, and the same Pontryagin numbers. The quotient is again a graded ring, which we call the Pontryagin–Poincaré ring $\mathcal{PP}_*$, graded by the dimension. By Corollary 2 there are no $\mathbb{Q}$-linear relations between the Betti and Pontryagin numbers. Therefore, by the classical result of Thom on the oriented bordism ring, we conclude

$$\mathcal{PP}_* \otimes \mathbb{Q} = (\mathcal{P}_* \otimes \mathbb{Q}) \oplus (\Omega^{SO}_* \otimes \mathbb{Q}),$$

where Ω^{SO}_* denotes the oriented bordism ring.

Theorem 15. *The forgetful homomorphism*

$$\tilde{F}\colon \mathcal{CH}_* \otimes \mathbb{Q} \longrightarrow \mathcal{PP}_* \otimes \mathbb{Q}$$

is surjective onto the even-degree part of $\mathcal{PP}_* \otimes \mathbb{Q}$. Its kernel is the ideal $\mathcal{JO}$ in $\mathcal{CH}_* \otimes \mathbb{Q}$ generated by differences of orientation-preservingly diffeomorphic smooth complex projective varieties.

Proof. Let E be an elliptic curve and $\beta_1 = \mathbb{C}P^1$, both considered as elements in $\mathcal{CH}_* \otimes \mathbb{Q}$. By Corollary 1 all $\mathbb{Q}$-linear combinations of Betti numbers in even dimensions are detected by polynomials in $\tilde{F}(E)$ and $\tilde{F}(\beta_1)$. These elements in $\mathcal{PP}_* \otimes \mathbb{Q}$ have trivial Pontryagin numbers. Using the same basis sequence β_i as in the previous proof, we see that the images $\tilde{F}(\beta_i)$ have trivial Betti numbers if $i \geqslant 2$, but any non-trivial linear combination of Pontryagin numbers is detected by polynomials in the $\tilde{F}(\beta_i)$ with even i. This proves that the image of $\tilde{F}$ is the even-degree part of $\mathcal{PP}_* \otimes \mathbb{Q}$.

It is clear that $\mathcal{JO} \subset \ker(\tilde{F})$ since Betti and Pontryagin numbers are oriented diffeomorphism invariants. By definition, $\mathcal{JO}$ is a subideal of $\mathcal{J}$, which, by the previous theorem, equals $\ker(F)$.

Using the same basis sequence as in the previous proof, $\mathcal{J} = \ker(F)$ is the ideal generated by all β_i with $i \geqslant 2$. By [Kot12, Theorem 7], this basis sequence has the property that for odd $i \geqslant 3$ the elements β_i and $\beta_1 \cdot \beta_{i-1}$ are in $\mathcal{JO}$. We also know that, for all $i \geqslant 2$, $E \cdot \beta_i$ is a difference of diffeomorphic projective varieties. Since E admits orientation-reversing self-diffeomorphisms we have $E \cdot \beta_i \in \mathcal{JO}$.

By the proof of surjectivity of $\tilde{F}$ onto the even-degree part of $\mathcal{PP}_* \otimes \mathbb{Q}$, no non-trivial polynomial in the β_i with i even can be in $\ker(\tilde{F})$. Thus $\ker(\tilde{F})$ is the ideal generated by the residue classes of the β_i with odd $i \geqslant 3$, and by the $\beta_1 \cdot \beta_j$ and $E \cdot \beta_j$ with j even. All these generators are in $\mathcal{JO}$, and so $\ker(\tilde{F}) \subset \mathcal{JO}$. This completes the proof. □

Remark 5. In Theorems 14 and 15 we worked over $\mathbb{Q}$ in order to be able to use the special basis sequences β_n for the unitary bordism ring constructed in [Kot12]. For $n \geqslant 5$ we could use instead certain generators for $\Omega^U_* \otimes \mathbb{Z}[\frac{1}{2}]$ constructed in [Sch, Proposition 4.4]. A generator β_2 with all the required properties, that would also work after inverting only 2, was obtained in the proof of Theorem 10 above, but in degrees $n = 3$ or 4 we do not have any alternative generators. Checking the numerical factors in [Kot12, Proposition 15], it turns out that the β_3 used in [Kot12] and in the above proofs works for $\Omega^U_* \otimes \mathbb{Z}[\frac{1}{2}]$, but the β_4 used there requires one to invert 3, in addition to inverting 2. Therefore, Theorems 14 and 15 are true for $\mathcal{CH}_* \otimes \mathbb{Z}[\frac{1}{6}]$.

We can finally prove Theorem 4.

Proof of Theorem 4. The vector space dual to $\mathcal{CH}_n \otimes \mathbb{Q}$ is made up of $\mathbb{Q}$-linear combinations of Hodge and Chern numbers, modulo the linear combinations of the $\chi_p - \mathrm{Td}_p$, and modulo the implicit Kähler symmetries. If a linear form on $\mathcal{CH}_n \otimes \mathbb{Q}$ defines an unoriented homeomorphism invariant, or an unoriented diffeomorphism invariant in dimension $n \geqslant 3$, then by Theorem 14 the corresponding homomorphism φ factors through F, and so reduces to a combination of Betti numbers. Conversely, linear combinations of Betti numbers are of course homeomorphism-invariant. This completes the proof of the second statement.

By Theorem 15 a linear form on $\mathcal{CH}_n \otimes \mathbb{Q}$ that defines an oriented diffeomorphism invariant factors through $\tilde{F}$, and therefore reduces to a combination of Betti and Pontryagin numbers, which make up the linear forms on $\mathcal{PP}_{2n} \otimes \mathbb{Q}$. Conversely, these linear combinations are invariant under orientation-preserving diffeomorphisms, and even under orientation-preserving homeomorphisms by a result of Novikov [Nov65]. This completes the proof of the first statement. □

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